

ASYMPTOTIC FOR A SEMILINEAR HYPERBOLIC EQUATION WITH ASYMPTOTICALLY VANISHING DAMPING TERM, CONVEX POTENTIAL, AND INTEGRABLE SOURCE

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Abstract: We investigate the long time behavior of solutions to semilinear hyperbolic equation on the forme:

$$(E_\alpha) \quad u''(t) + \gamma(t)u'(t) + Au(t) + f(u(t)) = g(t), \quad t \geq 0,$$

where A is a self-adjoint nonnegative operator, f a function which derives from a convex function, and γ a nonnegative function which behaviors, for t large enough, as $\frac{K}{t^\alpha}$ with $K > 0$ and $\alpha \in [0, 1[$. We obtain sufficient conditions on the source term $g(t)$, ensuring the weak or the strong convergence of any solution $u(t)$ of (E_α) as $t \rightarrow +\infty$ to a solution of the stationary equation $Av + f(v) = 0$ if one exists.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let H be a real Hilbert space with inner product and norm respectively denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$. V is an other Hilbert space continuously and densely embedded in H . Let V' the dual space of V . By identifying H with its dual space, we have $V \hookrightarrow H \hookrightarrow V'$. We recall the following important relation that will be used repeatedly in the sequel:

$$(1.1) \quad \langle v, w \rangle_{V', V} = \langle v, w \rangle \quad \forall (v, w) \in H \times V.$$

In this paper, we investigate the long time behavior as $t \rightarrow +\infty$ of solutions u to the following second order semi-linear hyperbolic equation:

$$(E_\alpha) \quad u''(t) + \gamma(t)u'(t) + Au(t) + f(u(t)) = g(t), \quad t \geq 0,$$

where the damping term γ is a function in $W_{loc}^{1,1}(\mathbb{R}^+, \mathbb{R}^+)$ which behaviors like $\frac{K}{t^\alpha}$ for some $K > 0$, $\alpha \in [0, 1[$, and t large enough. Precisely, we assume that there exist $K > 0$, $t_0 \geq 0$ and $\alpha \in [0, 1[$ such that:

$$(1.2) \quad \gamma(t) \geq \frac{K}{(1+t)^\alpha} \quad \forall t \geq t_0,$$

$$(1.3) \quad ((1+t)^\alpha \gamma(t))' \leq 0 \text{ a.e. } t \geq t_0.$$

The operator $A : V \rightarrow V'$ is linear and continuous. We suppose that the associated bilinear form $a : V \times V \rightarrow \mathbb{R}$ defined by

$$a(v, w) = \langle Av, w \rangle_{V', V}$$

is symmetric, positive and satisfies the semi-coercivity property:

$$\exists \lambda \geq 0, \mu > 0 : a(v, v) + \lambda |v|^2 \geq \mu \|v\|_V^2 \quad \forall v \in V.$$

We assume that the function $f : V \rightarrow V'$ is a continuous function which derives from a C^1 convex function $F : V \rightarrow \mathbb{R}$ in the sense:

$$(1.4) \quad \forall u, v \in V, F'(u)(v) = \langle f(u), v \rangle_{V', V}.$$

Let us consider the function $\Phi : V \rightarrow \mathbb{R}$ defined by:

$$\Phi(v) = \frac{1}{2}a(v, v) + F(v).$$

It is clear that Φ is a C^1 convex function and for all $v \in V$, $\nabla \Phi(v) = Av + f(v)$. We assume that the set

$$\arg \min \Phi = \{v \in V : \Phi(v) = \min_V \Phi := \Phi^*\},$$

which coincides with the set $\{v \in V : Av + f(v) = 0\}$, is nonempty. Last, we suppose that function $g : \mathbb{R}^+ \rightarrow H$ belongs to the space $L^1(\mathbb{R}^+, H)$.

In this paper, we assume the existence of a global solution u to Eq. (E_α) in the class

$$(1.5) \quad W_{loc}^{2,1}(\mathbb{R}^+, H) \cap W_{loc}^{1,1}(\mathbb{R}^+, V),$$

and we focus our attention on its asymptotic behavior as t goes to infinity. Before setting our main theorems, let us first recall some previous results. In the pioneer paper [1], Alvarez considered the case where $V = H$, the damping term γ is a non negative constant and the source g is equal to 0. He proved that every solution $u(t)$ converges weakly to a minimizer of the function Φ . Moreover, he showed that the convergence is strong if the function Φ is even or the interior of $\arg \min \Phi$ is not empty. In [7], Haraux and Jendoubi extended the weak convergence result of Alvarez to the case where the source term is in the space $L^1(\mathbb{R}^+, H)$. Recently, Cabot and Frankel [6] studied Eq. (E_α) where $g = 0$ and $\gamma(t)$ behaviors at infinity like $\frac{K}{t^\alpha}$ with $K > 0$ and $\alpha \in]0, 1[$. They proved that every bounded solution converges weakly toward a critical point of Φ . In a very recent work [9], the second author of the present paper improved the result of Cabot and Frankel by getting rid of the supplementary hypothesis on the boundedness of the solution. In [8], it was proved that the main convergence result of Cabot and Frankel remains true if the source term g satisfies the condition $\int_0^{+\infty} (1+t) |g(t)| dt < \infty$. The first purpose of the present paper is to improve this result. In fact, we prove that the convergence holds under the weaker and optimal condition

$$(1.6) \quad \int_0^{+\infty} (1+t)^\alpha |g(t)| dt < \infty.$$

Precisely, we establish the following result.

Theorem 1.1. *Assume that $\int_0^{+\infty} (1+t)^\alpha |g(t)| dt < \infty$. Let u be a solution to Eq. (E_α) in the class (1.5). If $u \in L^\infty(\mathbb{R}^+, H)$, then $u(t)$ converges weakly in V as $t \rightarrow +\infty$ toward some element of $\arg \min \Phi$. Moreover, the energy function*

$$(1.7) \quad \mathcal{E}(t) := \frac{1}{2} |u'(t)|^2 + \Phi(u(t)) - \Phi^*$$

satisfies $\mathcal{E}(t) = o(t^{-2\alpha})$ as $t \rightarrow +\infty$.

In the next theorem, we prove that we can get rid of the hypothesis on the boundedness of the solution by adding a second condition on the source term. This theorem generalizes the main result of [9].

Theorem 1.2. *Assume that $\int_0^{+\infty} (1+t)^\alpha |g(t)| dt < \infty$ and $\int_0^{+\infty} (1+t)^{3\alpha} |g(t)|^2 dt < \infty$. Let u be a solution to Eq. (E_α) in the class (1.5). Then $u \in L^\infty(\mathbb{R}^+, H)$ and, therefore, we have the same conclusion as in Theorem 1.1.*

Our two last main results concern the strong convergence of the solution when the potential function Φ is even or the interior of the set $\arg \min \Phi$ is nonempty.

Theorem 1.3. *Assume that the function Φ is even, $\int_0^{+\infty} (1+t)^\alpha |g(t)| dt < \infty$, and $\int_0^{+\infty} (1+t)^{2\alpha+1} |g(t)|^2 dt < \infty$. Let u be a solution to Eq. (E_α) in the class (1.5). Then there exists $u_\infty \in \arg \min \Phi$ such that $u(t) \rightarrow u_\infty$ strongly in V as $t \rightarrow +\infty$.*

Theorem 1.4. *Assume that*

$$(1.8) \quad \forall \varepsilon > 0 \exists C_\varepsilon > 0 : \forall v \in V, |v| \leq C_\varepsilon \|v\|_{V'} + \varepsilon \|v\|_V.$$

Assume moreover that the interior of the set $\arg \min \Phi$ with respect of the strong topology of V is not empty. Let u be a solution to Eq. (E_α) in the class (1.5). If $\int_0^{+\infty} (1+t)^\alpha |g(t)| dt < \infty$ and $u \in L^\infty(\mathbb{R}^+, H)$, then $u(t)$ converges strongly in V as $t \rightarrow +\infty$ to some element of $\arg \min \Phi$.

Remark 1.1. *The property (1.8) is equivalent to the following topological property: every sequence bounded in V and strongly convergent in V' converges strongly in H . Hence if there exist $C > 0$ and $\theta \in]0, 1[$ such that:*

$$(1.9) \quad \forall v \in V, |v| \leq C \|v\|_{V'}^{1-\theta} \|v\|_V^\theta,$$

then (1.8) is satisfied.

Remark 1.2. *In a very recent work [2], Attouch and Chbani have considered the second order differential equation*

$$(E_1) \quad u''(t) + \frac{K}{t} u'(t) + \nabla \Phi(u(t)) = g(t),$$

where $K > 0$, $\Phi : H \rightarrow \mathbb{R}$ is a smooth convex function, and $g \in L^1(\mathbb{R}^+, H)$. In Eq. (E_1) the damping term $\gamma(t) = \frac{K}{t}$ satisfies (1.2) and (1.3) with $\alpha = 1$. Hence (E_1) can be considered as a limit case of Eq. (E_α) . In [2], the authors supposed that the source term g satisfies only the optimal condition $\int_0^{+\infty} (1+t) |g(t)| dt < \infty$ corresponding to (1.6) with $\alpha = 1$, and they proved that if $K > 3$ and $\arg \min \Phi \neq \emptyset$ then every solution to (E_1) converges weakly to a minimizer of the function Φ and strongly if in addition Φ is even or the interior of $\arg \min \Phi$ is nonempty. It would be interesting to know whether our main results remain true if we suppose that the function g satisfies only the optimal condition (1.6) and without assuming that the solution u belongs to the space $L^\infty(\mathbb{R}^+, H)$.

Remark 1.3. *A typical example of Eq. (E_α) is the following nonlinear damped wave equation:*

$$u_{tt} + \gamma(t)u_t - \Delta u + f(u) = g \text{ on } \Omega \times]0, +\infty[,$$

with the Dirchlet boundary condition:

$$u = 0 \text{ on } \partial\Omega \times]0, +\infty[,$$

where Ω is a bounded open subset of \mathbb{R}^N , $g \in L^1([0, +\infty[, L^2(\Omega))$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nondecreasing function which satisfies

$$|f(s)| \leq C(1 + |s|)^m \quad \forall s \in \mathbb{R},$$

where C and m are nonnegative constants with $m \leq \frac{N}{N-2}$ if $N \geq 3$. Here $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $a(v, w) = \int_\Omega \nabla v \nabla w dx$, and F is the function defined on $H_0^1(\Omega)$ by:

$$F(v) = \int_\Omega \int_0^{v(x)} f(s) ds dx.$$

Using Sobolev's inequalities, one can easily verify that the function $v \mapsto f(v)$ is continuous from $H_0^1(\Omega)$ to $L^2(\Omega)$ and F is a C^1 convex function which satisfies the property (1.4), in fact

$$\forall v, w \in H_0^1(\Omega), F'(v)(w) = \int_{\Omega} f(v(x))w(x)dx.$$

Finally, since $L^2(\Omega) = [H^{-1}(\Omega), H_0^1(\Omega)]_{\frac{1}{2}}$, The property (1.9) is satisfied with $\theta = \frac{1}{2}$.

2. PRELIMINARY RESULTS

In this section we will prove some important preliminary results which will be very useful to prove the main theorems in the next section.

Proposition 2.1. *let u be a solution to Eq. (E_{α}) in the class (1.5). Assume that there exists $\nu \in [0, 1 + \alpha[$ such that: $\int_0^{+\infty} (1+t)^{\frac{\nu}{2}} |g(t)| dt < \infty$. Assume moreover that $u \in L^{\infty}(\mathbb{R}^+, H)$ or $\int_0^{+\infty} (1+t)^{\nu+\alpha} |g(t)|^2 dt < \infty$. Then*

$$\int_0^{+\infty} (1+t)^{\nu-\alpha} |u'(t)|^2 dt < \infty,$$

and the energy function \mathcal{E} , given by (1.7), satisfies $\mathcal{E}(t) = o(t^{-\nu})$ as $t \rightarrow +\infty$.

Proof. The proof of this proposition makes use of a modified version of a method introduced by Cabot et Frankel in [6] and recently developed in [9]. Let $\bar{u} \in \arg \min \Phi$ and define the function $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $p(t) = \frac{1}{2} |u(t) - \bar{u}|^2$. Since u is in the class (1.5), the function p belongs to the space $W_{loc}^{2,1}(\mathbb{R}^+, \mathbb{R}^+)$ and satisfies almost everywhere on \mathbb{R}^+

$$\begin{aligned} p''(t) + \gamma(t)p'(t) &= |u'(t)|^2 + \langle \nabla \Phi(u(t)), \bar{u} - u(t) \rangle + \langle g(t), u(t) - \bar{u} \rangle \\ &= |u'(t)|^2 + \langle \nabla \Phi(u(t)), \bar{u} - u(t) \rangle_{V', V} + \langle g(t), u(t) - \bar{u} \rangle \\ &\leq |u'(t)|^2 + \Phi(\bar{u}) - \Phi(u(t)) + |g(t)| \sqrt{2p(t)} \\ (2.1) \quad &= \frac{3}{2} |u'(t)|^2 - \mathcal{E}(t) + |g(t)| \sqrt{2p(t)}, \end{aligned}$$

where we have used (1.1) and the convexity inequality $\Phi(\bar{u}) \geq \Phi(u(t)) + \langle \nabla \Phi(u(t)), \bar{u} - u(t) \rangle_{V', V}$. On the other hand the energy function \mathcal{E} belongs to $W_{loc}^{1,1}(\mathbb{R}^+, \mathbb{R})$ and satisfies for almost every $t \geq 0$,

$$\begin{aligned} \mathcal{E}'(t) &= \langle u''(t), u'(t) \rangle + \langle \nabla \Phi(u(t)), u'(t) \rangle_{V', V} \\ &= \langle u''(t), u'(t) \rangle + \langle \nabla \Phi(u(t)), u'(t) \rangle \\ (2.2) \quad &= -\gamma(t) |u'|^2 + \langle g(t), u'(t) \rangle. \end{aligned}$$

For every $r \in \mathbb{R}$, we define the function λ_r on \mathbb{R}^+ by $\lambda_r(t) = (1+t)^r$. In view of (2.2),

$$(2.3) \quad (\lambda_{\nu} \mathcal{E})' = \lambda'_{\nu} \mathcal{E} - \lambda_{\nu} \gamma |u'|^2 + \lambda_{\nu} \langle g, u' \rangle.$$

Hence,

$$(2.4) \quad \lambda_\nu \gamma |u'|^2 \leq \lambda'_\nu \mathcal{E} - (\lambda_\nu \mathcal{E})' + \lambda_{\frac{\nu}{2}} |g| \sqrt{2\lambda_\nu \mathcal{E}}.$$

Since γ satisfies (1.2) with $\alpha < 1$, $\lambda'_\nu(t) |u'(t)|^2 = o(\lambda_\nu(t) \gamma(t) |u'(t)|^2)$ as $t \rightarrow +\infty$. Then there exists $t_1 \geq t_0$ such that

$$(2.5) \quad \frac{3}{2} \lambda'_\nu(t) |u'(t)|^2 \leq \frac{1}{2} \lambda_\nu(t) \gamma(t) |u'(t)|^2 \quad \text{a.e. } t \geq t_1.$$

Thus, by multiplying the inequality (2.1) by $\lambda'_\nu(t)$ and using (2.4)-(2.5), we obtain

$$\frac{1}{2} \lambda'_\nu \mathcal{E} + \frac{1}{2} (\lambda_\nu \mathcal{E})' \leq -\lambda'_\nu p'' - \lambda'_\nu \gamma p' + \lambda'_\nu |g| \sqrt{2p} + \frac{1}{2} \lambda_{\frac{\nu}{2}} |g| \sqrt{2\lambda_\nu \mathcal{E}},$$

almost everywhere on $[t_1, \infty[$. Integrating this last inequality between t_1 and $t \geq t_1$, we get after integrations by parts

$$(2.6) \quad \frac{1}{2} \int_{t_1}^t \lambda'_\nu \mathcal{E} ds + \frac{1}{2} (\lambda_\nu \mathcal{E})(t) \leq C_0 + A(t) + B(t) + C(t),$$

where

$$\begin{aligned} C_0 &= \frac{1}{2} (\lambda_\nu \mathcal{E})(t_1) + (\lambda'_\nu p')(t_1) - (\lambda''_\nu p)(t_1) + (\lambda'_\nu \gamma p)(t_1), \\ A(t) &= -(\lambda'_\nu p')(t) + (\lambda''_\nu p)(t) - (\lambda'_\nu \gamma p)(t), \\ B(t) &= \int_{t_1}^t (-\lambda_\nu^{(3)} + (\lambda'_\nu \gamma)') p + \lambda'_\nu |g| \sqrt{2p} ds, \\ C(t) &= \int_{t_1}^t \lambda_{\frac{\nu}{2}} |g| \sqrt{\lambda_\nu \mathcal{E}} ds. \end{aligned}$$

Let us estimate separately $A(t)$, $B(t)$, and $C(t)$. Firstly, by using the fact that $\sqrt{\lambda_\nu \mathcal{E}} \leq 1 + \lambda_\nu \mathcal{E}$, we get

$$(2.7) \quad C(t) \leq \int_0^{+\infty} (1+s)^{\frac{\nu}{2}} |g(s)| ds + \int_{t_1}^t \lambda_{\frac{\nu}{2}} |g| \lambda_\nu \mathcal{E} ds.$$

On the other hand, in view of (1.2)

$$\begin{aligned} A(t) &\leq \lambda'_\nu(t) |\langle u'(t), u(t) - \bar{u} \rangle| - \nu [K - (\nu - 1)(1+t)^{\alpha-1}] (1+t)^{\nu-\alpha-1} p(t) \\ &\leq 2\lambda'_\nu(t) \sqrt{\mathcal{E}(t)} \sqrt{p(t)} - \nu [K - (\nu - 1)(1+t)^{\alpha-1}] (1+t)^{\nu-\alpha-1} p(t). \end{aligned}$$

Therefore, since $\alpha < 1$, there exists $t_2 \geq t_1$ such that for every $t \geq t_2$,

$$A(t) \leq 2\lambda'_\nu(t) \sqrt{\mathcal{E}(t)} \sqrt{p(t)} - \frac{\nu K}{2} (1+t)^{\nu-\alpha-1} p(t).$$

Using now the elementary inequality

$$(2.8) \quad \forall a > 0 \quad \forall b, x \in \mathbb{R}, \quad -ax^2 + bx \leq \frac{b^2}{4a}$$

with $x = \sqrt{p(t)}$, we get

$$A(t) \leq \frac{2\nu}{K}(1+t)^{\nu+\alpha-1}\mathcal{E}(t) \quad \forall t \geq t_2.$$

Using once again the fact that $\alpha < 1$, we infer the existence of $t_3 \geq t_2$ such that

$$(2.9) \quad A(t) \leq \frac{1}{4}\lambda_\nu(t)\mathcal{E}(t) \quad \forall t \geq t_3.$$

Let us now prove that the function B is bounded. To this end we first notice that, thanks to (1.2)-(1.3), we have for almost every $t \geq t_1$

$$\begin{aligned} -\lambda_\nu^{(3)}(t) + (\lambda'_\nu \gamma)'(t) &\leq -\lambda_\nu^{(3)}(t) + \lambda''_\nu \gamma - \alpha \lambda'_\nu(t) \frac{\gamma(t)}{(1+t)} \\ &\leq -\lambda_\nu^{(3)}(t) - \nu K(1+\alpha-\nu)(1+t)^{\nu-2-a}. \end{aligned}$$

Since $\nu < 1 + \alpha$ and $\alpha < 1$, there exists $t_4 \geq t_3$ such that for almost every $t \geq t_4$,

$$(2.10) \quad -\lambda_\nu^{(3)}(t) + (\lambda'_\nu \gamma)'(t) \leq -\mu(1+t)^{\nu-2-a},$$

where $\mu = \frac{\nu K(1+\alpha-\nu)}{2} > 0$. Therefore, if $u \in L^\infty(\mathbb{R}^+, H)$ then for every $t \geq t_4$ we have

$$\begin{aligned} B(t) &\leq B(t_4) + \sqrt{\sup_{t \geq 0} 2p(t)} \int_0^{+\infty} \lambda'_\nu |g| dt \\ &\leq B(t_4) + \nu \sqrt{\sup_{t \geq 0} 2p(t)} \int_0^{+\infty} (1+t)^{\frac{\nu}{2}} |g| dt \end{aligned}$$

Let us now examine the boundedness of the function B under the other hypothesis $\int_0^{+\infty} (1+t)^{\nu+\alpha} |g(t)|^2 dt < \infty$. By using (2.10) and the inequality (2.8) with $x = \sqrt{p(t)}$ we easily get that for every $t \geq t_4$

$$\begin{aligned} B(t) &\leq B(t_4) + \frac{2\nu^2}{\mu} \int_{t_4}^t (1+s)^{\nu+\alpha} |g(s)|^2 dt \\ &\leq B(t_4) + \frac{2\nu^2}{\mu} \int_0^{+\infty} (1+s)^{\nu+\alpha} |g(s)|^2 dt. \end{aligned}$$

Coming back to (2.6) and using the estimates (2.7)-(2.9) and the boundedness of the function B , we infer the existence of a constant $C_1 \geq 0$ such that for every $t \geq t_4$,

$$\frac{1}{2} \int_{t_1}^t \lambda'_\nu \mathcal{E} ds + \frac{1}{4} (\lambda_\nu \mathcal{E})(t) \leq C_1 + \int_{t_1}^t \lambda_{\frac{\nu}{2}} |g| \lambda_\nu \mathcal{E} dt.$$

Therefore, by applying Gronwall's inequality we first get that $\sup_{t \geq t_1} \lambda_\nu(t) \mathcal{E}(t) < +\infty$ and then we deduce that $\int_{t_1}^{+\infty} \lambda'_\nu(t) \mathcal{E}(t) dt < +\infty$. Recalling that the energy function \mathcal{E} is continuous and hence locally bounded on \mathbb{R}^+ , we infer that

$$(2.11) \quad \int_0^{+\infty} \lambda'_\nu(t) \mathcal{E}(t) dt < +\infty$$

and

$$(2.12) \quad \sup_{t \geq 0} \lambda_\nu(t) \mathcal{E}(t) < +\infty.$$

Hence by using the equality (2.3) we obtain

$$\begin{aligned} \int_0^{+\infty} [(\lambda_\nu \mathcal{E})']_+ dt &\leq \int_0^{+\infty} \lambda'_\nu \mathcal{E} dt + \sqrt{\sup_{t \geq 0} 2\lambda_\nu(t) \mathcal{E}(t)} \int_0^{+\infty} \lambda_{\frac{\nu}{2}} |g| dt \\ &< +\infty, \end{aligned}$$

where $[(\lambda_\nu \mathcal{E})']_+$ is the positive part of $(\lambda_\nu \mathcal{E})'$. The last inequality implies that $\lambda_\nu(t) \mathcal{E}(t)$ converges as t goes to $+\infty$ to some real number m . If $m \neq 0$ then $\lambda'_\nu(t) \mathcal{E}(t) = \frac{\lambda_\nu(t) \mathcal{E}(t)}{\nu(1+t)} \sim \frac{m}{\nu(1+t)}$ as $t \rightarrow +\infty$ which contradicts the result (2.11). Thus $m = 0$ and therefore $\mathcal{E}(t) = o(t^{-\nu})$ as $t \rightarrow +\infty$. Finally, by using the inequality (2.4), we obtain

$$\int_0^{+\infty} \lambda_\nu \gamma |u'|^2 dt \leq \int_0^{+\infty} \lambda'_\nu \mathcal{E} dt + \mathcal{E}(0) + \sqrt{\sup_{t \geq 0} 2\lambda_\nu(t) \mathcal{E}(t)} \int_0^{+\infty} \lambda_{\frac{\nu}{2}} |g| dt.$$

In view of (2.11) and (2.12), the right hand side of the previous inequality is finite, then thanks to the hypothesis (1.2) we conclude that

$$\int_0^{+\infty} (1+t)^{\nu-\alpha} |u'(t)|^2 dt < +\infty$$

as desired. \square

Proposition 2.2. *Let u be a solution to Eq. (E_α) . Assume that the integrals $\int_0^{+\infty} (1+t)^\alpha |g(t)| dt$ and $\int_0^{+\infty} (1+t)^\alpha |u'(t)|^2 dt$ are finite and $\Phi(u(t)) \rightarrow \Phi^*$ as $t \rightarrow +\infty$. Then $u(t)$ converges weakly in V as $t \rightarrow +\infty$ toward some element u_∞ of $\arg \min \Phi$.*

The proof of this proposition repose on the classical Opial's lemma [10] (see [3] for a simple proof) and an elementary lemma which will be also used to prove Theorem 1.3 and Theorem 1.4. Let us first recall Opial's lemma.

Lemma 2.1 (Opial's lemma). *Let $x : [t_0, +\infty[\rightarrow \mathcal{H}$. Assume that there exists a nonempty subset S of \mathcal{H} such that:*

- i: *If $t_n \rightarrow +\infty$ and $x(t_n) \rightharpoonup x$ weakly in \mathcal{H} , then $x \in S$.*
- ii: *For every $z \in S$, $\lim_{t \rightarrow +\infty} \|x(t) - z\|$ exists.*

Then there exists $z_\infty \in S$ such that $x(t) \rightharpoonup z_\infty$ weakly in \mathcal{H} as $t \rightarrow +\infty$.

Lemma 2.2. *There exists $\tau_0 \geq 0$ such that for every $\tau \geq \tau_0$*

$$\int_\tau^{+\infty} e^{-\Gamma(t,\tau)} dt \leq \frac{2}{K} (1+\tau)^\alpha$$

where $\Gamma(t, \tau) = \int_\tau^t \gamma(s) ds$.

Proof. Let $\tau \geq t_0$. In view of (1.2),

$$\begin{aligned} \int_{\tau}^{+\infty} e^{-\Gamma(t,\tau)} dt &\leq \frac{1}{K} \int_{\tau}^{+\infty} (1+t)^{\alpha} \gamma(t) e^{-\Gamma(t,\tau)} dt \\ &= -\frac{1}{K} \int_{\tau}^{+\infty} (1+t)^{\alpha} \left(e^{-\Gamma(t,\tau)} \right)' dt \\ &= \frac{1}{K} (1+\tau)^{\alpha} + \frac{\alpha}{K} \int_{\tau}^{+\infty} (1+t)^{\alpha-1} e^{-\Gamma(t,\tau)} dt \\ &\leq \frac{1}{K} (1+\tau)^{\alpha} + \frac{\alpha}{K(1+\tau)^{1-\alpha}} \int_{\tau}^{+\infty} e^{-\Gamma(t,\tau)} dt. \end{aligned}$$

Hence to conclude we just have to choose τ_0 large enough such that $\frac{\alpha}{K(1+\tau_0)^{1-\alpha}} \leq \frac{1}{2}$. \square

Proof of Proposition 2.2 . Let us first prove that $u \in L^{\infty}(\mathbb{R}^+, V)$. Let $\bar{u} \in \arg \min \Phi$ and define, as in the proof of Proposition 2.1, the function $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $p(t) = \frac{1}{2} |u(t) - \bar{u}|^2$. This function belongs to the space $W_{loc}^{2,1}(\mathbb{R}^+, \mathbb{R}^+)$ and satisfies almost everywhere on \mathbb{R}^+ ,

$$\begin{aligned} p'' + \gamma p' &= |u'|^2 - \langle \nabla \Phi(u), u - \bar{u} \rangle + \langle g, u - \bar{u} \rangle \\ &= |u'|^2 - \langle \nabla \Phi(u) - \nabla \Phi(\bar{u}), u - \bar{u} \rangle_{V', V} + \langle g, u - \bar{u} \rangle \\ &\leq |u'|^2 + |g| \sqrt{2p}, \end{aligned}$$

where we have used the monotonicity of the operator $\nabla \Phi$. Therefore, for almost every $t \geq \tau_0$,

$$p'(t) \leq e^{-\Gamma(t,\tau_0)} p'(\tau_0) + \int_{\tau_0}^t e^{-\Gamma(t,s)} \rho(s) ds$$

where $\rho := |u'|^2 + |g| \sqrt{2p}$. Thus, by using the previous lemma and Fubini's theorem, we get for every $t \geq \tau_0$

$$\begin{aligned} \int_{\tau_0}^t [p'(\tau)]_+ d\tau &\leq \frac{2(1+\tau_0)^{\alpha}}{K} |p'(\tau_0)| + \frac{2}{K} \int_{\tau_0}^t (1+s)^{\alpha} \rho(s) ds \\ (2.13) \quad &\leq c_0 + \frac{2}{K} \int_{\tau_0}^t (1+s)^{\alpha} |g(s)| \sqrt{2p(s)} ds \end{aligned}$$

where $c_0 = \frac{2(1+\tau_0)^{\alpha}}{K} |p'(\tau_0)| + \frac{2}{K} \int_0^{+\infty} (1+s)^{\alpha} |u'(s)|^2 ds$ and $[p'(\tau)]_+$ is the positive part of $p'(\tau)$. Using now the inequalities $\sqrt{2p} \leq 1 + 2p$ and $p(t) \leq p(\tau_0) + \int_{\tau_0}^t [p'(\tau)]_+ d\tau$, we obtain

$$p(t) \leq c_1 + \frac{4}{K} \int_{\tau_0}^t (1+s)^{\alpha} |g(s)| p(s) ds, \quad \forall t \geq \tau_0,$$

with $c_1 = c_0 + p(\tau_0) + \frac{2}{K} \int_0^{+\infty} (1+s)^{\alpha} |g(s)| ds$. Hence, by applying Gronwall's inequality, we deduce that the function p is bounded which is equivalent to $u \in L^{\infty}(\mathbb{R}^+, H)$. Using now [6,

Remark 3.4], we obtain that $u \in L^\infty(\mathbb{R}^+, V)$. Coming back to the estimate (2.13), we infer that

$$\begin{aligned} \int_{\tau_0}^{+\infty} [p'(\tau)]_+ d\tau &\leq c_0 + \frac{2}{K} \int_0^{+\infty} (1+s)^\alpha |g(s)| ds \sqrt{\sup_{t \geq 0} 2p(t)} \\ &< +\infty \end{aligned}$$

which implies that $\lim_{t \rightarrow +\infty} p(t)$ and therefore $\lim_{t \rightarrow +\infty} |u(t) - \bar{u}|$ exist. Now, let $\bar{x} \in H$ such that there exists a sequence $(t_n)_n$ of positive real numbers tending to $+\infty$ such that $u(t_n)$ converges weakly in H to \bar{x} . Since $u \in L^\infty(\mathbb{R}^+, V)$, $u(t_n)$ converges weakly also in the space V to the same element \bar{x} . Using now the weak lower semi-continuity of the continuous and convex function Φ , we deduce that $\Phi^* = \liminf \Phi(u(t_n)) \leq \Phi(\bar{x})$. Thus $\bar{x} \in \arg \min \Phi$. Therefore, applying Opial's lemma with $S = \arg \min \Phi$ ensures that $u(t)$ converges weakly in H as $t \rightarrow +\infty$ to some element of $\arg \min \Phi$. Recalling that $u \in L^\infty(\mathbb{R}^+, V)$, we conclude that this weak convergence holds also in the space V . \square

3. PROOF OF THE MAIN RESULTS

This section is devoted to the proof of our main theorems. Let us first notice that Theorem 1.1 and Theorem 1.2 follow immediately from Proposition 2.1 (with $\nu = 2\alpha$) and Proposition 2.2. So we are left to prove Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. The proof is adapted from the original method introduced by Bruck [5] for the steepest descent method and used by Alvarez [1] for the heavy ball with friction system. Since $2\alpha + 1 \geq 3\alpha$, then in view of Theorem 1.2, Proposition 2.1, and Proposition 2.2, $u(t)$ converges weakly in V to some $u_\infty \in \arg \min \Phi$, and $\int_0^{+\infty} (1+t)^\alpha |u'(t)|^2 dt < \infty$. Let $\tau \geq \tau_0$ where τ_0 is the real defined in Lemma 2.2. We define the function q on the interval $[\tau_0, \tau]$ by:

$$q(t) = |u(t)|^2 - |u(\tau)|^2 - \frac{1}{2} |u(t) - u(\tau)|^2.$$

The function q belongs to the space $W^{2,1}([\tau_0, \tau], \mathbb{R})$ and satisfies almost everywhere

$$(3.1) \quad q'(t) = \langle u'(t), u(t) + u(\tau) \rangle$$

$$(3.2) \quad q''(t) = |u'(t)|^2 + \langle u''(t), u(t) + u(\tau) \rangle.$$

Combining this two equalities, we obtain

$$\begin{aligned} q''(t) + \gamma(t)q'(t) &= |u'(t)|^2 + \langle \nabla \Phi(u), -u(\tau) - u(t) \rangle_{V', V} + \langle g(t), u(t) + u(\tau) \rangle \\ &\leq |u'(t)|^2 + \Phi(-u(\tau)) - \Phi(u(t)) + 2M |g(t)| \\ &= |u'(t)|^2 + \Phi(u(\tau)) - \Phi(u(t)) + 2M |g(t)| \\ (3.3) \quad &= \frac{3}{2} |u'(t)|^2 + \tilde{\mathcal{E}}(\tau) - \tilde{\mathcal{E}}(t) + 2M |g(t)| + \int_t^\tau \frac{|g(s)|^2}{4\gamma(s)} ds \end{aligned}$$

where $M = \sup_{t \geq 0} |u(t)|$ and $\tilde{\mathcal{E}}$ is the modified energy function defined by:

$$\tilde{\mathcal{E}}(t) = \mathcal{E}(t) + \int_t^{+\infty} \frac{|g(s)|^2}{4\gamma(s)} ds,$$

where \mathcal{E} is the energy function given by (1.7). Using (2.2), we get

$$\begin{aligned} \tilde{\mathcal{E}}'(t) &= -\gamma(t) |u'(t)|^2 + \langle g(t), u'(t) \rangle - \frac{|g(t)|^2}{4\gamma(t)} \\ &\leq - \left(\sqrt{\gamma(t)} |u'(t)| - \frac{|g(t)|}{2\sqrt{\gamma(t)}} \right)^2. \end{aligned}$$

Therefore the function $\tilde{\mathcal{E}}$ is non increasing. Hence (3.3) and (1.2) yield

$$q''(t) + \gamma(t)q'(t) \leq \omega(t),$$

where

$$\omega(t) = \frac{3}{2} |u'(t)|^2 + 2M |g(t)| + \frac{1}{4K} \int_t^{+\infty} (1+s)^\alpha |g(s)|^2 ds.$$

Therefore, for almost every $t \in [\tau_0, \tau]$,

$$(3.4) \quad q'(t) \leq e^{-\Gamma(t, \tau_0)} |q'(\tau_0)| + \int_{\tau_0}^t e^{-\Gamma(t, s)} \omega(s) ds \equiv \kappa(t).$$

A simple calculation, using Fubini's theorem and Lemma 2.2, gives

$$\int_{\tau_0}^{+\infty} \kappa(t) dt \leq c_0 + \frac{2}{K} \int_{\tau_0}^{+\infty} (1+s)^\alpha \omega(s) ds$$

where $c_0 = \frac{2}{K} (1 + \tau_0)^\alpha |q'(\tau_0)|$. Using once again Fubini's theorem, we get

$$\int_{\tau_0}^{+\infty} (1+t)^\alpha \int_t^{+\infty} (1+s)^\alpha |g(s)|^2 ds dt \leq \frac{1}{\alpha+1} \int_{\tau_0}^{+\infty} (1+s)^{2\alpha+1} |g(s)|^2 ds.$$

Then we deduce that the integral $\int_{\tau_0}^{+\infty} (1+s)^\alpha \omega(s) ds$ is finite which implies

$$(3.5) \quad \int_{\tau_0}^{+\infty} \kappa(t) dt < +\infty.$$

Integrating now (3.4) between t and τ , with $\tau_0 \leq t \leq \tau$, we get

$$(3.6) \quad \frac{1}{2} |u(t) - u(\tau)|^2 \leq |u(t)|^2 - |u(\tau)|^2 + \int_t^\tau \kappa(s) ds.$$

In the proof of Proposition 2.2, we showed that $\lim_{t \rightarrow +\infty} |u(t) - \bar{u}|^2$ exists for all \bar{u} in $\arg \min \Phi$. But $0 \in \arg \min \Phi$ since Φ is convex and even, then $\lim_{t \rightarrow +\infty} |u(t)|^2$ exists. Therefore, (3.6) and (3.5) imply

$$|u(\tau) - u(t)| \rightarrow 0 \text{ as } t, \tau \rightarrow +\infty.$$

Thus, in view of Cauchy criteria, $u(t)$ converges strongly in H as $t \rightarrow +\infty$. Therefore, by using [6, Corollary 3.6], we deduce that $u(t)$ converges strongly in V as $t \rightarrow +\infty$. Finally, since $u(t) \rightharpoonup u_\infty$ weakly in V , we conclude that $u(t) \rightarrow u_\infty$ strongly in V . \square

Proof of Theorem 1.4. The main idea of this proof is inspired from the paper [4].

By assumption, there exists $x^* \in \arg \min \Phi$ and $r > 0$ such that for all v in the unit Ball $B_V(0, 1)$ of V we have $\nabla \Phi(x^* + rv) = 0$. Therefore the monotonicity of $\nabla \Phi$ implies that for every $x \in V$, $\langle \nabla \Phi(x), x - x^* - rv \rangle_{V', V} \geq 0$ which yields that $\langle \nabla \Phi(x), v \rangle_{V', V} \leq \frac{1}{r} \langle \nabla \Phi(x), x - x^* \rangle_{V', V}$. Hence by taking the supremum on $v \in B_V(0, 1)$, we get

$$(3.7) \quad \|\nabla \Phi(x)\|_{V'} \leq \frac{1}{r} \langle \nabla \Phi(x), x - x^* \rangle_{V', V}.$$

Let us now define the function $p(t) = \frac{1}{2} |u(t) - x^*|^2$. We already know that p satisfies the differential inequality

$$p''(t) + \gamma(t)p'(t) \leq |u'(t)|^2 - \langle \nabla \Phi(u(t)), u(t) - x^* \rangle_{V', V} + \langle g(t), u(t) - x^* \rangle.$$

Hence by using (3.7), we obtain

$$(3.8) \quad r \|\nabla \Phi(u(t))\|_{V'} \leq -p''(t) - \gamma(t)p'(t) + \sigma(t),$$

where $\sigma(t) = |u'(t)|^2 + |g(t)| \sup_{t \geq 0} |u(t) - x^*|$.

Recalling that in view Proposition 2.1, $\int_0^{+\infty} \lambda_\alpha(t) \sigma(t) dt < \infty$ where $\lambda_\alpha(t) = (1+t)^\alpha$. Hence, by multiplying (3.8) by $\lambda_\alpha(t)$ and integrating between t_0 and $\tau \geq t_0$, we get after integrations by parts and simplification

$$\begin{aligned} r \int_{t_0}^{\tau} \lambda_\alpha(t) \|\nabla \Phi(u(t))\|_{V'} dt &\leq C - \lambda_\alpha(\tau) p'(\tau) + \lambda'_\alpha(\tau) p(\tau) \\ &\quad - \underbrace{(\lambda_\alpha \gamma)(\tau) p(\tau)}_{\geq 0} + \int_{t_0}^{\tau} \underbrace{[(\lambda_\alpha \gamma)' - \lambda''_\alpha](t) p(t)}_{\leq 0} dt \end{aligned}$$

where C is a constant independent of τ . Since $\alpha < 1$ and $u \in L^\infty(\mathbb{R}^+, H)$, the integral $\int_{t_0}^{+\infty} |\lambda''_\alpha(t)| p(t) dt$ and the supremum $\sup_{\tau \geq t_0} \lambda'_\alpha(\tau) p(\tau)$ are finite. Moreover, from Proposition 2.1, $|u'(\tau)| = o(\tau^{-\alpha})$ as $\tau \rightarrow +\infty$, then

$$\sup_{\tau \geq t_0} \lambda_\alpha(\tau) |p'(\tau)| \leq \sup_{\tau \geq t_0} \lambda_\alpha(\tau) |u'(\tau)| |u(\tau) - x^*| < \infty.$$

Therefore, we conclude that

$$(3.9) \quad \int_{t_0}^{+\infty} \lambda_\alpha(t) \|\nabla \Phi(u(t))\|_{V'} dt < +\infty.$$

From Eq. (E $_\alpha$), we have

$$u''(t) + \gamma(t)u'(t) = g(t) - \nabla \Phi(u(t))$$

Hence, by integrating this equation we get

$$(3.10) \quad u'(t) = e^{-\Gamma(t, \tau_0)} u'(\tau_0) + \int_{\tau_0}^t e^{-\Gamma(t, s)} [g(s) - \nabla \Phi(u(s))] ds,$$

for almost every $t \geq \tau_0$ where τ_0 is the real defined by Lemma 2.2. Up to replace τ_0 by $\tau'_0 > \tau_0$, we can assume that $u'(\tau_0) \in H$. Thus by applying Lemma 2.2 and Fubini's theorem to the equality (3.10), we obtain

$$\begin{aligned} \int_{\tau_0}^{+\infty} \|u'(t)\|_{V'} dt &\leq \frac{2}{K}(1+\tau_0)^\alpha \|u'(\tau_0)\|_{V'} + \frac{2}{K} \int_{\tau_0}^{+\infty} (1+s)^\alpha \|g(s)\|_{V'} \\ &\quad + \frac{2}{K} \int_{\tau_0}^{+\infty} (1+s)^\alpha \|\nabla \Phi(u(s))\|_{V'} ds. \end{aligned}$$

Hence $\int_{\tau_0}^{+\infty} \|u'(t)\|_{V'} dt < +\infty$ thanks to the continuous injection $H \hookrightarrow V'$, the hypothesis on g , and the estimate (3.9). Thus we deduce that $u(t)$ converges strongly in V' as $t \rightarrow +\infty$ to some u_∞ . Recalling that, in view of Theorem 1.1, $u \in L^\infty(\mathbb{R}^+, V)$ and applying the hypothesis (1.8), we infer that $u(t) \rightarrow u_\infty$ strongly in H , which in view of [6, Corollary 3.6] implies that $u(t)$ converges strongly to u_∞ in V . Finally, Theorem 1.1 ensures that $u_\infty \in \arg \min \Phi$. The proof is completed. \square

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